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Long Range Forces and Supersymmetric Bound States

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Abstract

We consider the long range forces between two BPS particles on the Coulomb branch of $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric gauge theories. The $1/r$ potential is unambiguously fixed, even at strong coupling, by the moduli dependence of central charges supported by the BPS states. The effective Coulombic coupling vanishes on marginal stability curves, while sign changes on crossing these curves explain the restructuring of the spectrum of composite BPS states. This restructuring proceeds via the delocalization of the composite state on approach to the curve of marginal stability. Therefore the spectrum of BPS states can be inferred by analyzing the submanifolds of the moduli space where the long range potential is attractive. This method also allows us to find certain non-BPS bound states and their stability domains. As examples, we consider the dissociation of the W boson and higher charge dyons at strong coupling in $\mathcal{N}=2$ SU(2) SYM, quark-monopole bound states in $\mathcal{N}=2$ SYM with one flavor, and composite dyons in $\mathcal{N}=2$ SU(3) SYM.

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I. INTRODUCTION

Theories with extended supersymmetry admit special shortened massive multiplets which preserve some fraction of supersymmetry. In particular, $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric gauge theories possess Bogomol'nyi-Prasad-Sommerfield (BPS) states $|\mathbf{n}\rangle$ which along the Coulomb branch are ground states in sectors with a given set $\mathbf{n} = \{\mathbf{n}_i\}$ of conserved $U(1)$ charges. As shown in [1, 2], BPS states preserve some fraction of supersymmetry because they support at least one central charge \mathcal{Z} of the superalgebra, $\mathcal{Z}|\mathbf{n}\rangle = \mathcal{Z}_{\mathbf{n}}|\mathbf{n}\rangle$. Moreover, BPS states inherit special stability properties from the constraint that their masses are fixed in terms of their central charges $M_{\mathbf{n}} = |\mathcal{Z}_{\mathbf{n}}|$. Such a state is thus stable with respect to decay into two particles, $|\mathbf{n}_1\rangle$ and $|\mathbf{n}_2\rangle$, with $\mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2$. Indeed, since the central charges are linear in $\{\mathbf{n}_i\}$, $M_{\mathbf{n}} = |\mathcal{Z}_{\mathbf{n}_1} + \mathcal{Z}_{\mathbf{n}_2}| \leq M_{\mathbf{n}_1} + M_{\mathbf{n}_2}$. However, the central charges generically depend on the moduli (and/or parameters) in the theory, and so there can be special submanifolds where a BPS state is only marginally stable, i.e. $M_{\mathbf{n}} = M_{\mathbf{n}_1} + M_{\mathbf{n}_2}$. These *curves of marginal stability* (CMS) in the moduli (and/or parameter) space are then of interest as they are the only regimes where a discontinuous change in the spectrum of BPS states is possible.

The existence of marginal stability submanifolds is quite generic in theories where the superalgebra can support central charges. They have been observed in two dimensional models [3], in the particle spectrum of $\mathcal{N}=2$ and $\mathcal{N}=4$ SYM [4–9], and in related Type IIB string junctions [10, 11]. More generally, they arise in the consideration of the D-brane spectrum in string compactifications on manifolds with nontrivial cycles [12]. CMS curves may also arise for extended objects such as BPS domain walls [13, 14] in theories without extended supersymmetry.

While the existence and position of CMS curves is straightforwardly deduced from the BPS mass formula and the moduli dependence of the central charges $\mathcal{Z}_{\mathbf{n}}$, it is not immediately apparent whether a discontinuity in the spectrum does occur, and if so on which side of the CMS do the BPS states actually exist, or indeed whether they exist at all. However, Bilal and Ferrari [6, 7, 9] were able to show that, for $\mathcal{N}=2$ theories with gauge group $SU(2)$, a full reconstruction of the BPS spectrum accounting for its discontinuities on the CMS can in fact be deduced from the moduli dependence of the central charges. The derivation is not dynamical, it is based on continuity, symmetries, and knowledge of the singularities present in the Seiberg-Witten [4] low energy solution. A somewhat more dynamical construction of the spectrum was subsequently made via an examination of Type IIB string junctions [10, 11] in appropriate backgrounds.

In this paper we will show that the dynamics which governs the formation of composite BPS states, and describes the corresponding discontinuities of the spectrum, is quite straightforward. In particular, there are long range Coulombic forces between BPS particles due to massless electrostatic, magnetostatic, and scalar exchanges, and it is the sign of this long range potential which determines the presence or otherwise of bound states. Moreover, this potential is unambiguously fixed by the moduli dependence of the central charges and proves to be calculable in all regions of the moduli space.

The importance of this potential relies on the central idea of our work together with Shifman and Voloshin [15], which is that in the near CMS region “composite” BPS particles can be viewed as weakly bound states of “primary” BPS particles, and moreover that these

composite states *delocalize* on approach to the CMS. In 3+1D this delocalization follows from the structure of the BPS mass formula and the constraint that the Coulombic attraction must vanish on the CMS. Generically the effective Coulombic coupling changes sign upon crossing the CMS, and thus attraction changes to repulsion and the “composite” BPS state is removed from the physical spectrum. The work presented here generalizes the analysis of [15] to strong coupling.

Delocalization near the CMS allows us to treat the BPS constituents as point-like charges interacting via long range forces and described by supersymmetric quantum mechanics (SQM) [16]. This approximation is valid when the binding energy $E_{\text{bind}} = M_{\mathbf{n}_1} + M_{\mathbf{n}_2} - M_{\mathbf{n}}$ is much less than the constituent masses $M_{\mathbf{n}_1}, M_{\mathbf{n}_2}$. Crucially, this constraint can be satisfied for any CMS curve, including those at strong coupling, provided we keep away from the singularities where the BPS constituents become massless, $M_{\mathbf{n}_1, \mathbf{n}_2} = 0$. Therefore, by evaluating the long range potential and observing in which regions of the moduli space it is attractive or repulsive, we can deduce the spectrum of BPS and also non-BPS bound states composed from BPS constituents. While it might seem rather remarkable that such an approximation makes sense in the strong coupling region, the crucial point is that since the constituent states delocalize on approach to the CMS, the interaction between constituents can be made arbitrarily weak, and thus the approximation arbitrarily good by moving closer to the CMS itself.

The physical picture is that as a composite state is moved closer to its CMS curve it behaves like a weakly bound Coulomb system, and its constituents dissociate on the CMS itself. Notice that in terms of the scattering amplitude for the $|\mathbf{n}_1\rangle$ and $|\mathbf{n}_2\rangle$ particles one still observes continuity on crossing the CMS in the following sense. In the complex energy plane the pole singularity associated with the bound state moves to the beginning of the continuum cut when the moduli approach the CMS. After crossing the CMS the singularity goes back along the same line but on the second sheet of the Riemann surface. A singularity of this kind, close to the continuum and on the second sheet is known as a virtual state (or virtual level). This situation is quite distinct from the formation of a resonance, and for this reason we prefer to call the phenomenon on crossing the CMS a “dissociation” (or delocalization) into primary particles rather than a “decay”, the term often used in the literature.

The discussion above was framed in terms of $\mathcal{N}=2$ supersymmetry with one complex central charge \mathcal{Z} . However, by considering a massless $\mathcal{N}=2$ hypermultiplet in the adjoint representation we can include the $\mathcal{N}=4$ case. It is well known that duality [17] predicts a BPS spectrum for $\mathcal{N}=4$ SYM which is considerably richer than that of $\mathcal{N}=2$ SYM, e.g. for gauge group $\text{SU}(2)$ there are BPS states with magnetic charge two (and nonvanishing co-prime electric charge) present in $\mathcal{N}=4$ SYM, which are not in the $\mathcal{N}=2$ BPS spectrum. Making use of the Coulombic potential, we can understand this behavior as follows: breaking $\mathcal{N}=4$ to $\mathcal{N}=2$ actually preserves bound states with magnetic charge two, in the semiclassical domain of the moduli space, but these states are *not* BPS states of $\mathcal{N}=2$. Thus, the total BPS plus non-BPS spectrum behaves continuously under breaking to $\mathcal{N}=2$. Moreover, we find that these non-BPS states possess a new marginal stability curve that we call the “non-BPS CMS”, or “nCMS”, and are not present inside a certain strong coupling submanifold of the moduli space bounded in part by this curve. This result about the spectrum of certain non-BPS states of higher magnetic charge in $\mathcal{N}=2$ $\text{SU}(2)$ SYM is in

agreement with the arguments of Bergman [18] made using the string junction construction.

The layout of the paper is as follows. In Sec. II we describe the calculation of the long range potential between two BPS states due to massless exchange. We then apply this potential to a number of examples illustrating how it can be used to deduce features of the BPS and non-BPS bound state spectrum in $\mathcal{N}=4$ and $\mathcal{N}=2$ SYM. In Sec. III, we consider pure $\mathcal{N}=4$ and $\mathcal{N}=2$ SYM with gauge group $SU(2)$, reproducing the expected BPS spectrum at weak and strong coupling, and deducing the presence of many non-BPS states in $\mathcal{N}=2$ SYM. In Sec. IV, we consider $\mathcal{N}=2$ SYM with gauge group $SU(2)$ and one massive flavor, which provides a convenient example for considering the effect of the Argyres-Douglas point [19] on the behavior of the potential in the strong coupling region. Quark-monopole bound states were considered in similar theories in [9], and at weak coupling in [20]. In Sec. V, we generalize the construction to gauge group $SU(3)$. In this system CMS curves extend to the weak coupling region, and we study the restructuring of the spectrum of dyons in the semiclassical regime on breaking $\mathcal{N}=4$ to $\mathcal{N}=2$. Composite dyons in this theory have recently been considered in the moduli space approximation in [21–28, 15, 29]. Finally, in Sec. VI, we finish with some concluding remarks and directions for future work.

While this work was being written up we received an interesting preprint [30] by Argyres and Narayan which also discusses the discontinuities of the BPS spectrum in $\mathcal{N}=2$ and $\mathcal{N}=4$ SYM. Our work shares with [30] the general picture [15] of loosely bound BPS states near the CMS. However, the approach of Argyres and Narayan is technically different from ours, as they describe equilibrium configurations rather than the interactions, and their construction makes a bridge to the string web picture (see also [31]). It would be interesting to explore the possible connections between these approaches in more detail.

II. THE BPS SPECTRUM AND LONG RANGE POTENTIALS

In this section we discuss in detail the form of the long range potential between BPS sources, and obtain a convenient representation near the CMS.

A. Generic Constraints

We begin by recalling some of the general constraints imposed on long range interactions between BPS states in SYM, as discussed in [15]. First, as mentioned above, it is clear that if we consider the dynamics of two BPS particles with masses $M_{\mathbf{n}_1}$ and $M_{\mathbf{n}_2}$ sufficiently near the CMS for the composite state $|\mathbf{n}\rangle = |\mathbf{n}_1 + \mathbf{n}_2\rangle$ with mass $M_{\mathbf{n}}$, then the binding energy E_{bind} (and, consequently, the kinetic energy) can be made much smaller than the masses $M_{\mathbf{n}_1}$ and $M_{\mathbf{n}_2}$ of the primary states themselves. i.e. by moving near the CMS we have,

$$\epsilon = \frac{E_{\text{bind}}}{M_r} \ll 1, \quad E_{\text{bind}} = M_{\mathbf{n}_1} + M_{\mathbf{n}_2} - M_{\mathbf{n}}, \quad M_r = \frac{M_{\mathbf{n}_1} M_{\mathbf{n}_2}}{M_{\mathbf{n}_1} + M_{\mathbf{n}_2}}, \quad (1)$$

where M_r is the reduced mass of the two particles. This is a restriction that we need to keep away from points in moduli space where the BPS states are massless. However, this is not a significant constraint as such submanifolds are of higher co-dimension than

the CMS curves. Therefore we can always consider a regime near the CMS where the dynamics describing the composite state is manifestly nonrelativistic, i.e. where $\epsilon \ll 1$. In this regime we can legitimately ignore the full microscopic theory and study the effective (supersymmetric) quantum mechanics associated with the collective coordinates of the BPS states. It is important that this argument relies only on having $\epsilon \ll 1$ and is therefore quite general; it applies whether or not the underlying theory is strongly or weakly coupled.

Moreover, in 3+1D this point of view can be pushed one stage further since the notion of marginal stability addressed here can be rephrased at the quantum mechanical level as a question about the presence or otherwise of bound states. The form of the BPS mass formula indicates that as we move near a CMS curve, bound states are formed via an attractive coupling that can be made arbitrarily small. In contrast with 1+1 and 2+1 dimensions where any arbitrarily small attraction is sufficient to form a bound state, in 3+1D only long range potentials can form a bound state for arbitrarily small coupling, and thus we deduce that such forces must be present. In other words, bound state formation near the CMS depends only on long range forces, and is therefore insensitive to detailed issues about short range interactions between the core structures of the BPS states. Consequently, we can simplify the effective description by treating the primary states as point-like particles interacting only via long range forces. This is clearly an abstraction but these arguments indicate that it is quite sufficient for answering questions about the presence of bound states.

The consistency of this argument can be verified by noting that Coulombic systems associated with an attractive $1/r$ -type potential,

$$V(r) = -\frac{f}{4\pi r}, \quad (2)$$

in the limit where the coupling f becomes small possess towers of closely spaced bound states, only the lowest of which can be BPS saturated. In contrast, we know from the BPS mass formula that on the CMS the lowest level in the tower must reach the continuum. This can only happen if the effective coupling f , which is a function of the moduli, vanishes on the CMS.

A corollary of these constraints is that the composite BPS configuration, viewed as a bound state of primary constituents, must delocalize on approach to the CMS. An example of such delocalization at the classical level of static equilibrium configurations is provided by the semiclassical system of two dyons with charges under different $U(1)$ subgroups of the Cartan subalgebra in $\mathcal{N}=2$ SYM with gauge group $SU(3)$ [21,24,25,28,29], as discussed from this standpoint in [15]. In this system, besides the Coulombic potential (2) with a coupling f which vanishes on the CMS, there is also a subleading repulsive $1/r^2$ potential which does not vanish on the CMS. This leads to a diverging equilibrium separation on approach to the CMS, which becomes infinite on the CMS itself.

With this information in hand, it follows that the discontinuity in the spectrum on crossing a CMS can be inferred purely from the leading Coulombic potential between constituent states. The rest of this section is devoted to constructing this potential for a generic $\mathcal{N}=2$ theory (we include $\mathcal{N}=4$ SYM by allowing for the presence of a massless adjoint hypermultiplet).

B. Evaluating the Long Range Potential

Recall first of all that the mass of a BPS state is given by the absolute value of the corresponding central charge \mathcal{Z}

$$M_{\mathbf{n}} = |\mathcal{Z}_{\mathbf{n}}|, \quad \mathcal{Z} |\mathbf{n}\rangle = \mathcal{Z}_{\mathbf{n}} |\mathbf{n}\rangle. \quad (3)$$

The central charge \mathcal{Z} arising in the $\mathcal{N}=2$ supersymmetry algebra is a linear combination of conserved $U(1)$ charges $\{Q_i\}$ with coefficients c^i which are functions of the vacuum moduli u_k ,

$$\mathcal{Z} = \sqrt{2} \sum_i Q_i c^i(u). \quad (4)$$

We label the state $|\mathbf{n}\rangle$ with a set of parameters $\mathbf{n} = \{n_i\}$ which are the charges of the state with respect to the $\{Q_i\}$,

$$Q_i |\mathbf{n}\rangle = n_i |\mathbf{n}\rangle. \quad (5)$$

Thus,

$$\mathcal{Z}_{\mathbf{n}} = \sqrt{2} \sum_i n_i c^i(u). \quad (6)$$

For $\mathcal{N}=2$ theories with gauge group $SU(N)$ maximally broken to $U(1)^{N-1}$, the charges \mathbf{n} we deal with can be represented in the following form $\mathbf{n} = \{n_E^a, n_{Ma}, s^f\}$, where the index $a = 1, \dots, N-1$ refers to a suitable basis in the Cartan subalgebra. In this set, n_E^a and n_{Ma} refer respectively to the lattice of electric charges and the dual lattice of magnetic charges. In our conventions, n_E^a and n_{Ma} are integers or half-integers¹. Note that half-integer electric charges arise in the presence of fundamental matter, while half-integer magnetic charges appear in a related way, as discussed below, for higher rank gauge groups. Finally, s^f are integral flavor charges associated with $U(1)$ symmetries of the additional hypermultiplets. The central charge may then be written in the form

$$\mathcal{Z}_{\{n_E^a, n_{Ma}, s^f\}} = \sqrt{2} [n_E^a a_a(u) + n_{Ma} a_D^a(u)] + s^f m_f, \quad (7)$$

where a_a and a_D^a are functions of the vacuum moduli u_k which are defined via powers of the adjoint scalar field Φ as $u_k = \langle \text{Tr } \Phi^k \rangle$ + products of lower order Casimirs, and m_f is the mass of the f -th hypermultiplet.

The low-energy effective Lagrangian for the $\mathcal{N}=2$ SYM massless fields, written in terms of $\mathcal{N}=1$ superfields, has the following form [32],

$$\mathcal{L}_{\text{eff}} = \frac{1}{8\pi} \text{Im} \int d^2\theta W_a W_D^a + \frac{1}{4\pi} \text{Im} \int d^2\theta d^2\bar{\theta} \bar{A}_a A_D^a \quad (8)$$

¹In this respect our conventions differ from Seiberg and Witten [4], who modified the definition of the corresponding vev to make n_E^a , for example, integral. Another difference in conventions is the $\sqrt{2}$ in the definition of the central charge \mathcal{Z} .

where $\langle A_a \rangle = a_a(u)$, $\langle A_D^a \rangle = a_D^a(u)$, and W_D and $A_D^a(A)$ are defined by the prepotential $\mathcal{F}(A)$,

$$W_D^a = \tau^{ab}(A) W_b, \quad A_D^a(A) = \frac{\partial \mathcal{F}(A)}{\partial A_a}, \quad \tau^{ab}(A) = \frac{\partial^2 \mathcal{F}(A)}{\partial A_a \partial A_b}. \quad (9)$$

Thus far we have not specified a basis in the Cartan subalgebra. However, the integrality properties of the charges discussed above are assured by choosing a basis in which the index $a = 1, \dots, N-1$ effectively enumerates the simple roots $\{\beta_a\}$ (for the electric charges) and the fundamental weights $\{\omega^a\}$ (for the magnetic charges) of the algebra (see e.g. [33]). This structure is natural recalling that the simple roots and fundamental weights are orthonormal², $\omega^a \cdot \beta_b = \delta_b^a$, and generate dual lattices. They are related via the Cartan matrix, which for $SU(N)$ takes the form $C_{ab} = \beta_a \cdot \beta_b$ and in our normalization $C_{aa} = 1$ and $C_{aa\pm 1} = -1/2$. The transition between, say, the electric charge lattice and its dual is accomplished via $\beta_a = C_{ab} \omega^b$.

This index convention is well adapted to the duality structure of the low energy effective theory. In the semiclassical regime the coupling matrix τ^{ab} defined in Eq. (9) is proportional to the Cartan matrix,

$$\tau^{ab} \rightarrow \tau_0 C^{ab}, \quad \tau_0 = \frac{4\pi i}{g_0^2} + \frac{\theta_0}{2\pi}, \quad (10)$$

and we can write $a_a = \langle \Phi \cdot \beta_a \rangle$ and $a_D^a = \tau^{ab} a_b$. It is then helpful to define magnetic charges with a lifted index $n_M^a = C^{ab} n_{Mb}$ which are referred to the basis of simple roots [34, 35]. This leads to the conventional semiclassical form for the central charge,

$$\mathcal{Z}_{\{n_E^a, n_M^a, s^f\}}^{\text{cl}} = \sqrt{2} [n_E^a + \tau_0 n_M^a] a_a + s^f m_f. \quad (11)$$

Note that while the magnetic charges n_M^a are integral, they lead to charges $n_{Ma} = C_{ab} n_M^b$ on the dual lattice which contain half-integral components.

The sources for the massless fields described by (8) are massive hypermultiplets, each described by a pair X, \tilde{X} of $\mathcal{N}=1$ chiral superfields. The low energy effective Lagrangian for the hypermultiplets, which includes their interaction with the massless fields, has the form

$$\mathcal{L}_{\text{hyp}} = \sum_X \left\{ \int d^2\theta d^2\bar{\theta} \left[\bar{X} e^{n_E^a V_a} e^{n_{Ma} V_D^a} X + \tilde{X} e^{-n_E^a V_a} e^{-n_{Ma} V_D^a} \tilde{X} \right] + 2 \text{Re} \int d^2\theta \tilde{X} \mathcal{Z}_X(A) X \right\}, \quad (12)$$

where n_E^a, n_{Ma} are the electric and magnetic charges of the field X , the superfield V_D is dual to V , and $\mathcal{Z}_X(A)$ is given by Eq. (7) in which \mathcal{Z} is viewed as a function of A_a .

The Lagrangians (8) and (12) together with the expression (7) for the central charge contain everything required to determine the long range interaction between two static BPS particles due to the massless exchanges shown schematically in Fig. 1. Indeed, \mathcal{L}_{hyp} describes the couplings between the hypermultiplets and the massless fields while \mathcal{L}_{eff} defines

²We will not distinguish roots and co-roots for $SU(N)$.

the massless propagators. Note that the point-particle approximation allows us to avoid any subtleties with considering states which are not mutually local within a field theoretic description. Strictly, we are calculating the classical energy for a system of heavy particles with electric and magnetic charges.

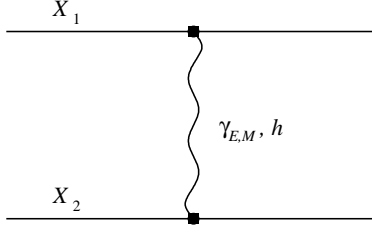


FIG. 1. The tree level massless exchanges leading to the long range potential between the BPS states X_1 and X_2 . $\gamma_{E,M}$ refers to the electrostatic and magnetostatic terms, while h denotes the scalar field quanta.

The $1/r$ potential is obtained as the sum of three terms,

$$V(r) = V_E(r) + V_M(r) + V_S(r), \quad (13)$$

referring to electrostatic V_E , magnetostatic V_M , and scalar V_S exchange, in correspondence with the contributions to the tree level diagram in Fig. 1 where γ_a and h_a are the intermediate massless particles. Let us start with the electrostatic interaction. When all magnetic charges n_{Ma} are zero we can straightforwardly read off its form from the low energy effective Lagrangians (8), (12). The part of the Lagrangian (8) describing the electric field is

$$\mathcal{L}_E = -\frac{1}{2} g^{ab} \vec{E}_a \vec{E}_b, \quad g^{ab} = \frac{\text{Im } \tau^{ab}}{4\pi}. \quad (14)$$

The electrostatic interaction V_E is then

$$V_E = \frac{1}{4\pi r} n_E^{(1)a} g_{ab} n_E^{(2)b}, \quad g_{ab} = \left[\frac{4\pi}{\text{Im } \tau} \right]_{ab}, \quad (15)$$

where g_{ab} is the inverse of the metric g^{ab} defined in Eq. (14).

From the dual description it is clear that when only magnetic charges are present the interaction is

$$V_M = \frac{1}{4\pi r} n_{Ma}^{(1)} g^{ab} n_{Mb}^{(2)}. \quad (16)$$

When both electric and magnetic charges are present, besides summing up the potentials (15) and (16), one also needs to account for the Witten effect [36], i.e. the electric charges n_{Ea} in Eq. (15) should be substituted by $n_{Ea} + n_M^b \text{Re } \tau_{ab}$. In terms of massless exchange, the Witten effect implies a mixing between the fields V_a and V_D^a . The mixing arises because the field V_D^a contains an electric part $\text{Re } \tau^{ab} V_b$ along with the magnetic part which does not mix with V_a . Thus, for the sum of V_E and V_M we get

$$V_E + V_M = \frac{1}{4\pi r} \text{Re} \left\{ \left(n_E^{(1)a} + n_{Mc}^{(1)} \tau^{ca} \right) g_{ab} \left(n_E^{(2)b} + \bar{\tau}^{bd} n_{Md}^{(2)} \right) \right\}, \quad (17)$$

where the bar denotes complex conjugation. Noting that $\partial\mathcal{Z}/\partial a_a = \sqrt{2}(n_E^a + n_{Mc}\tau^{ca})$ we can rewrite this in the form

$$V_E + V_M = \frac{1}{8\pi r} \text{Re} \left\{ g_{ab} \frac{\partial \mathcal{Z}_1}{\partial a_a} \frac{\partial \bar{\mathcal{Z}}_2}{\partial \bar{a}_b} \right\}. \quad (18)$$

The “effective charge” in this expression can be thought of as the dispersion between the two central charges \mathcal{Z}_1 and \mathcal{Z}_2 as measured via the Kähler metric.

We now turn to the scalar exchange potential which will be the object of interest for the remainder of this subsection. The non-supersymmetric scalar potential was first discussed from this point of view by Montonen and Olive [17] where it was used to verify the absence of static forces between two identical W bosons. It depends on the Yukawa couplings associated with interaction of the massless fields A^a with the BPS states X_i . Importantly, for BPS particles these couplings are fixed by supersymmetry and can be deduced by expansion of the corresponding central charges \mathcal{Z}_X . Indeed, these couplings follow from the superpotential term in \mathcal{L}_{hyp} (the last term in Eq. (12)). When A_a is substituted by its vacuum value $a_a(u)$ this term gives the mass for the field X . Introducing the deviations $H_a = A_a - a_a(u)$ from the vevs and expanding the central charge \mathcal{Z}_X ,

$$\mathcal{Z}_X(A) = \mathcal{Z}_X(a) + \frac{\partial \mathcal{Z}_X(a)}{\partial a_a} H_a + \dots, \quad (19)$$

we deduce the couplings of interest,

$$\begin{aligned} & 2 \text{Re} \left\{ \frac{\partial \mathcal{Z}_X(a)}{\partial a_a} \int d^2\theta H_a \tilde{X} X \right\} \\ &= -2 \text{Re} \left\{ \bar{\mathcal{Z}}_X(a) \frac{\partial \mathcal{Z}_X(a)}{\partial a_a} h_a \right\} (|x|^2 + |\tilde{x}|^2) + \text{fermionic terms}, \end{aligned} \quad (20)$$

where h_a , x and \tilde{x} are the bosonic components of H_a , X and \tilde{X} .

We also require the propagator for the deviations h_a . Expanding the second term in Eq. (8) we find, in the quadratic approximation, the Lagrangian for the h_a fields,

$$\mathcal{L}_h = g^{ab} \partial_\mu \bar{h}_a \partial^\mu h_b, \quad (21)$$

where the metric g^{ab} is defined in Eq. (14). The propagator for the scalars h_a is proportional to the inverse metric g_{ab} ,

$$\int d^4x e^{ikx} \langle T\{h_a(x), \bar{h}_b(0)\} \rangle = i \frac{g_{ab}}{k^2}. \quad (22)$$

With this information in hand, we can straightforwardly write down the static long range potential between the two BPS sources corresponding to the tree level scalar exchange in Fig. 1,

$$V_S(r) = -\frac{1}{8\pi r} \text{Re} \left\{ \left(g_{ab} \frac{\partial \mathcal{Z}_1}{\partial a_a} \frac{\partial \bar{\mathcal{Z}}_2}{\partial \bar{a}_b} \right) \left(\frac{\bar{\mathcal{Z}}_1 \mathcal{Z}_2}{|\mathcal{Z}_1| |\mathcal{Z}_2|} \right) \right\}. \quad (23)$$

We see that the scalar exchange differs from $V_E + V_M$, given in Eq. (18), only by presence of a phase factor naturally interpreted as the relative orientation of the two central charges. Furthermore, we observe that, accounting for the presence of the metric g_{ab} , this expression is reparametrization invariant with respect to the choice of the exchanged scalar field. In other words, our choice of that scalar as a_a was purely for convenience and the result would be the same if we had chosen any other scalar, e.g. deviations of moduli u_k , or a duality related combination such as a_D^a , which is an important consistency check. The use of a_a is nonetheless convenient because this is the field which lies in the same supermultiplet as the photon.

For the total Coulombic potential $V = V_E + V_M + V_S$, we find

$$\begin{aligned} V(r) &= \frac{1}{8\pi r} \operatorname{Re} \left\{ \left(g_{ab} \frac{\partial \mathcal{Z}_1}{\partial a_a} \frac{\partial \bar{\mathcal{Z}}_2}{\partial \bar{a}_b} \right) \left(1 - \frac{\bar{\mathcal{Z}}_1 \mathcal{Z}_2}{|\mathcal{Z}_1| |\mathcal{Z}_2|} \right) \right\} \\ &= \frac{1}{r} \operatorname{Re} \left\{ \left(n_E^{(1)a} + n_{Mc}^{(1)} \tau^{ca} \right) \left[\frac{1}{\operatorname{Im} \tau} \right]_{ab} \left(n_E^{(2)b} + \bar{\tau}^{bd} n_{Md}^{(2)} \right) \left(1 - \frac{\bar{\mathcal{Z}}_1 \mathcal{Z}_2}{|\mathcal{Z}_1| |\mathcal{Z}_2|} \right) \right\}. \end{aligned} \quad (24)$$

The first term in the final parentheses is due to $V_E + V_M$ while the relative phase between the central charges comes from V_S . The potential (24) constitutes the long range interaction between any two BPS particles. The generic feature which is immediately seen from Eq. (24) is that the potential vanishes on the CMS, where \mathcal{Z}_1 and \mathcal{Z}_2 are aligned, i.e. where $\operatorname{Im}(\bar{\mathcal{Z}}_1 \mathcal{Z}_2) = 0$ and $\operatorname{Re}(\bar{\mathcal{Z}}_1 \mathcal{Z}_2) > 0$. Let us introduce the angle of disalignment ω between the two central charges,

$$e^{i\omega} = \frac{\bar{\mathcal{Z}}_1 \mathcal{Z}_2}{|\mathcal{Z}_1| |\mathcal{Z}_2|}. \quad (25)$$

Then $V(r)$ can be rewritten as

$$\begin{aligned} V(r) &= \frac{1 - \cos \omega}{r} \left\{ \left(n_E^{(1)a} + n_{Mc}^{(1)} \operatorname{Re} \tau^{ca} \right) \left[\frac{1}{\operatorname{Im} \tau} \right]_{ab} \left(n_E^{(2)b} + \operatorname{Re} \tau^{bd} n_{Md}^{(2)} \right) + n_{Ma}^{(1)} \operatorname{Im} \tau^{ab} n_{Mb}^{(2)} \right\} \\ &\quad - \frac{\sin \omega}{r} \left(n_E^{(1)a} n_{Ma}^{(2)} - n_{Ma}^{(1)} n_E^{(2)a} \right), \end{aligned} \quad (26)$$

which provides a convenient form for consideration of the near-CMS regime.

C. The Potential in the Near-CMS Region

In the vicinity of the CMS, i.e. for $|\omega| \ll 1$, the last term in Eq. (26) is dominant and we arrive at a remarkably simple expression for the Coulombic potential in the near-CMS region,

$$\begin{aligned} V_{\text{CMS}} &= -\frac{\omega}{r} \left(n_E^{(1)a} n_{Ma}^{(2)} - n_{Ma}^{(1)} n_E^{(2)a} \right) + \mathcal{O}(\omega^2) \\ &= -\frac{1}{r} \operatorname{Im} \frac{\bar{\mathcal{Z}}_1 \mathcal{Z}_2}{|\mathcal{Z}_1| |\mathcal{Z}_2|} \left(n_E^{(1)a} n_{Ma}^{(2)} - n_{Ma}^{(1)} n_E^{(2)a} \right) + \mathcal{O}(\omega^2). \end{aligned} \quad (27)$$

This potential contains no explicit dependence on the couplings τ^{ab} , and thus on the Kähler metric, the dependence on moduli enters only via the angle of disalignment ω . The potential vanishes on the CMS where $\omega = 0$, while for existence of the composite $\mathbf{n}^{(1)} + \mathbf{n}^{(2)}$ BPS state the potential V_{CMS} should be attractive. It is clear that this will be the case on only one side of the CMS, and thus on the other side the constituents feel a long range repulsion and the BPS state ceases to exist.

At this point we should mention that the expression (27) is the near CMS form for a generic charge sector, where the product $(n_{Ea}^{(1)} n_M^{(2)a} - n_M^{(1)a} n_{Ea}^{(2)})$ is nonzero. In sectors for which this combination vanishes (e.g. if the constituents are purely electrically or purely magnetically charged) the total potential generically starts at $\mathcal{O}(\omega^2)$ as is apparent from Eq. (26).

In the following sections we will consider some illustrative examples which exhibit the basic phenomena of interest.

III. SPECTRUM IN SU(2) YANG-MILLS THEORY

In this section we use the long range potential to determine the spectrum of bound states for an SU(2) gauge group. Let us consider $\mathcal{N}=2$ SU(2) theory with one adjoint hypermultiplet. The expression for the central charge in this case has the form

$$\mathcal{Z}_{\{n_E, n_M, s\}} = \sqrt{2} [n_E a(u) + n_M a_D(u)] + s m_h, \quad (28)$$

where m_h is the mass of the hypermultiplet, s is the flavor charge associated with the hypermultiplet, and a and a_D are functions of the modulus $u = \langle \text{Tr } \Phi^2 \rangle$. The $\mathcal{N}=1$ chiral superfield Φ is the $\mathcal{N}=2$ partner of the photon, and its s -charge is zero. The hypermultiplet is described by two superfields Φ_s for which $s = \pm 1$.

This UV finite theory represents $\mathcal{N}=4$ SU(2) SYM at $m_h = 0$. When m_h is finite the theory flows in the infrared to $\mathcal{N}=2$ SU(2) SYM, where m_h plays the role of a UV cut off while the infrared scale Λ is

$$\Lambda^4 = 4 m_h^4 \exp(2\pi i \tau_0), \quad \tau_0 = \frac{4\pi i}{g_0^2} + \frac{\theta_0}{2\pi}, \quad (29)$$

where the normalization is chosen for convenience.

A. $\mathcal{N}=4$ SYM

Defining $\mathcal{N}=4$ SU(2) SYM as the limit $m_h \rightarrow 0$ implies that vevs of the hypermultiplet fields Φ_s must vanish. On the Coulomb branch of the $\mathcal{N}=4$ theory it is always possible to pass to such an orientation by global SU(4) rotations in the space of scalar fields. There is no running of the coupling and the vevs are given by their classical values $a = \sqrt{2u}$, $a_D = \tau_0 a$. The central charge (28) then takes the form (11),

$$\mathcal{Z}_{\{n_E, n_M\}} = \sqrt{2} a (n_E + n_M \tau_0). \quad (30)$$

The general expression (24) for the long range potential in this case results in

$$V(r) = -\frac{1 - \cos \omega}{r \operatorname{Im} \tau_0} \left| n_E^{(1)} + n_M^{(1)} \tau_0 \right| \left| n_E^{(2)} + n_M^{(2)} \tau_0 \right| \quad (31)$$

for the interaction between two BPS particles with generic electric and magnetic charges $\{n_E^{(i)}, n_M^{(i)}\}$. This expression is valid for any nonzero a , while the point $a = 0$ where all masses vanish should be excluded for the validity of our nonrelativistic approximation. We observe that the potential (31) vanishes if the central charges are aligned, i.e. $\omega = 0$, while it is attractive whenever they are misaligned.

In order to study the spectrum, let us start from the interaction of the particles $\pm\{1, 0\}$ and $\pm\{0, 1\}$. These are the lightest particles with nonvanishing electric and magnetic charges which guarantees their stability, and thus we will take these as “primary” constituents. The potential (31) leads to attraction in the channels $\pm\{1, 1\}$, $\pm\{1, -1\}$ but vanishes in the $\pm\{2, 0\}$, $\pm\{0, 2\}$ channels. Therefore the potential leads to bound states with quantum numbers $\pm\{1, 1\}$ and $\pm\{1, -1\}$. Although we are unable to use the nonrelativistic approximation to calculate the binding energy – it is of the same order as the reduced mass – we know that the ground states in the $\pm\{1, 1\}$, $\pm\{1, -1\}$ channels are indeed BPS saturated from the corresponding dyon solutions at weak coupling.

The vanishing of the $1/r$ potential in the $\pm\{2, 0\}$ and $\pm\{0, 2\}$ channels raises the question of whether localized threshold states in these channels could appear due to subleading corrections. However, there are compelling arguments against such a scenario. Firstly, attractive long range potentials, i.e. of $\mathcal{O}(1/r^2)$, are forbidden by the constraint that the mass cannot pass below the BPS bound, as such potentials would always lead to a nonzero binding energy. Secondly, bound states formed via short range interactions are inconsistent with the moduli space formulation of multi-monopole dynamics, valid at weak coupling, where in these channels the relative separation is an exact modulus, and bound states are not expected [37].

Knowledge of the long range potential between the two primary states allows us to deduce the full spectrum of stable bound states in the following way. Given that interactions between the primary states are either attractive, i.e. between $\pm\{1, 0\}$ and $\pm\{0, 1\}$, or vanish, we can ask whether the constituents of a given composite can be arranged into two or more subgroups which do not interact. If the answer is positive, then the two subgroups can dissociate without any cost in energy and consequently a bound state is not formed by the $1/r$ potential. On the other hand if the answer is negative, the constituents cannot be separated and we have a bound state.

For a generic configuration with charges $\{n_E, n_M\}$, it follows that we can arrange the constituents into k non-interacting subgroups only if the charges have a common divisor k , i.e. they have the form $\{n_E, n_M\} = k \{n'_E, n'_M\}$ for some integer k . Under the assumption that the bound states we find are indeed BPS, which we can verify in the semiclassical region for a restricted subset of charges, we then deduce that the stable BPS states form the set $\{n_E, n_M\}$ where n_E and n_M are co-prime. Thus, we have obtained a spectrum of BPS states which is in agreement with known semiclassical results [37], and indeed with the predictions of $\mathrm{SL}(2, \mathbf{Z})$ duality [17, 37, 38]. We will show that similar arguments can be applied to deduce the spectrum in the $\mathcal{N}=2$ theory.

B. Bound states and dissociation at strong coupling in $\mathcal{N}=2$ SU(2)

In $\mathcal{N}=4$ SU(2) SYM, since the gauge coupling is a marginal parameter, we can choose to stay in the weak coupling regime. However, as we have emphasized, our approach is not limited to weak coupling as we can make use of the Seiberg-Witten solution for $\mathcal{N}=2$ SU(2) SYM to determine the long range interactions even when the microscopic gauge system is strongly coupled.

Recall firstly that in the strong coupling region the BPS mass formula cannot be specified globally due to the presence of cuts in the relations $a(u)$ and $a_D(u)$ involving the global moduli space coordinate u . This implies that some states, while unique, require a different specification of charges on each side of the cut (see e.g. [6] for a careful analysis of this issue). We will avoid this technical complication by always restricting our attention to the upper half plane in the moduli space, $\text{Im}(u) > 0$, as parametrized by u , thus avoiding the cuts which can be chosen to lie on the real axis.

To begin with we consider the interaction between two BPS particles with generic electric and magnetic charges $\{n_E^{(i)}, n_M^{(i)}\}$. The central charges of these states are

$$\mathcal{Z}_i = \sqrt{2} \left[n_E^{(i)} a(u) + n_M^{(i)} a_D(u) \right]. \quad (32)$$

The angle of disalignment ω is given by

$$e^{i\omega} = \frac{n_E^{(1)} + n_M^{(1)} (a_D^*/a^*)}{\left| n_E^{(1)} + n_M^{(1)} (a_D/a) \right|} \frac{n_E^{(2)} + n_M^{(2)} (a_D/a)}{\left| n_E^{(2)} + n_M^{(2)} (a_D/a) \right|}. \quad (33)$$

The long range interaction vanishes at $\omega = 0$, i.e. when the central charges \mathcal{Z}_1 and \mathcal{Z}_2 are aligned,

$$\text{Im}(\bar{\mathcal{Z}}_1 \mathcal{Z}_2) = 0, \quad \text{Re}(\bar{\mathcal{Z}}_1 \mathcal{Z}_2) > 0. \quad (34)$$

These conditions are satisfied all over the moduli space when $\{n_E^{(1)}, n_M^{(1)}\}$ and $\{n_E^{(2)}, n_M^{(2)}\}$ are parallel. This means that when the total electric (n_E) and magnetic (n_M) charges are *not* co-prime the situation is the same as it was in $\mathcal{N}=4$: the state dissociates into noninteracting BPS particles. However, if $\{n_E^{(1)}, n_M^{(1)}\}$ and $\{n_E^{(2)}, n_M^{(2)}\}$ are not parallel the phase ω becomes moduli dependent. From the first condition in (34) it follows that the modulus u is on the CMS curve defined by

$$\text{Im}(a^*(u) a_D(u)) = 0, \quad (35)$$

which is sketched in Fig. 2.

Indeed, for $|\omega| \ll 1$, we have from Eq. (33) that

$$\omega = \text{Im}(a^* a_D) \frac{n_E^{(1)} n_M^{(2)} - n_M^{(1)} n_E^{(2)}}{\left| n_E^{(1)} a + n_M^{(1)} a_D \right| \left| n_E^{(2)} a + n_M^{(2)} a_D \right|} + \mathcal{O}(\omega^2). \quad (36)$$

This expansion is valid only when the second condition in Eq. (34) is fulfilled, i.e. for

from (-1) at $u = -\Lambda^2$ (the dyon $\{1, 1\}$ is massless at this point) to 0 at $u = \Lambda^2$ (with a massless monopole $\{0, 1\}$). It is clear then that the states $\pm\{0, 1\}$ and $\pm\{1, 1\}$ are stable BPS particles over all of the moduli space. Thus in passing to strong coupling the W boson is replaced by the $\{1, 1\}$ dyon as the lightest state carrying electric charge, which should then be chosen as a primary particle.

The W boson must then be understood as a composite configuration and indeed we can consider the two particle system of $\{0, -1\}$ and $\{1, 1\}$ which has the appropriate quantum numbers to form W^+ . The condition (37) is satisfied in this case, so we arrive at the scenario described above: the W boson exists as a bound state only in the exterior region of the CMS, its wavefunction swells on approach to the CMS, the system becomes delocalized, and the bound state does not exist in the interior region. Note that this picture is not in contradiction with the point-like nature of the W boson in the fundamental theory. At large moduli, $|u| \gg |\Lambda|^2$, we have a point-like W boson with a small admixture of the dyon pair. As we move close to the CMS this pair becomes dominant, and the picture of the W boson as a bound state becomes appropriate.

We can build up the spectrum of BPS states following the procedure outlined for $\mathcal{N}=4$ SYM. The new ingredient here is that we take $\{0, 1\}$ and $\{1, 1\}$ as our primary constituents (in the upper half-plane). Moreover, the potential between these states may be either attractive or repulsive depending on the choice of moduli. Thus, for a given composite configuration, we can ask whether its possible to arrange the primary constituents into subgroups in such a way that the energy is minimized when, for example, two of these subgroups are at infinite separation. If so, then no bound state is formed, while if the answer is negative, then we are guaranteed that a bound state is formed with those charges at that point in the moduli space.

Lets consider first the possible composite configurations with magnetic charge ± 1 . Following the above procedure, making use of the potential (38), we obtain all the dyons $\pm\{k, 1\}$ with $k = 2, 3, \dots$ and $k = -1, -2, \dots$ as bound states in the exterior region to the CMS, while we find no bound states in the interior region. The fact that these states are BPS is easily verified semiclassically from the moduli space formulation, where they arise via quantization of the electric charge, which is conjugate to one of the monopole moduli. Together with the W^\pm bosons these BPS particles form the well-known stable BPS spectrum [4, 6] in the exterior region of the CMS.

C. Non-BPS bound states in $\mathcal{N}=2$ SYM

We now proceed further and consider composite configurations with magnetic charge two. It is convenient to start with the simplest example corresponding to a system with charges $\{1, 2\}$. We take states $\{0, 1\}$ and $\{1, 1\}$ as primary in the upper half plane. The condition (37) is not satisfied in this case, i.e. $\text{Re}(\bar{\mathcal{Z}}^{(1)} \mathcal{Z}^{(2)}) < 0$ on the curve (35). This means that $\omega = \pi$ and we cannot use the potential (38), as there is no CMS for this state according to our definition above. However, we can still study the full Coulombic potential,

$$V_{\{1,2\}}(r) = \frac{1}{r \text{Im } \tau} \text{Re} \left\{ \tau (1 + \bar{\tau}) \left(1 - \frac{a_D^* (a + a_D)}{|a_D| |a + a_D|} \right) \right\}, \quad (39)$$

where

$$\tau(u) = \frac{\partial a_D}{\partial a} = \frac{4\pi i}{g^2(u)} + \frac{\theta(u)}{2\pi}. \quad (40)$$

In the weak coupling region where $|u| \gg |\Lambda|^2$, we have $a_D = \tau a$ and this potential matches the $\mathcal{N}=4$ potential given in Eq. (31) with the substitution of τ_0 by τ . The potential is attractive and the bound state we deal with is a descendent of the $\mathcal{N}=4$ BPS state with magnetic charge two.

At first sight the appearance of states with magnetic charge 2 in $\mathcal{N}=2$ SYM looks puzzling, as it is well known that such states do not exist as BPS states in this theory, i.e. one can argue following Bilal and Ferrari [6, 7] that were such a state to exist it would become massless on the CMS curve when $a_D/a = -1/2$ which contradicts the Seiberg-Witten solution. The resolution is simple: the bound state we have found is *not* an $\mathcal{N}=2$ BPS state. At large u this state can be viewed as a BPS state in $\mathcal{N}=4$. However, the part of $\mathcal{N}=4$ SUSY preserved by this state corresponds to generators carrying a nonzero value of s , i.e. involving fields from the $\mathcal{N}=2$ adjoint hypermultiplet. Thus, the supermultiplet we are dealing with here is short in terms of $\mathcal{N}=4$ supersymmetry but it is a “long” representation of $\mathcal{N}=2$ supersymmetry. In the classical approximation, valid for large u , the bosonic configuration satisfies first order differential equations and the soliton mass is given by the central charge. However, quantum corrections lift the mass $M_{\{1,2\}}$ above the BPS bound $|\mathcal{Z}_{\{1,2\}}|$. Nonetheless, the presence of an attractive potential means that this mass is still less than the sum of the constituent masses,

$$|\mathcal{Z}_{\{1,2\}}| < M_{\{1,2\}}^{\text{non-BPS}} < |\mathcal{Z}_{\{0,1\}}| + |\mathcal{Z}_{\{1,1\}}|. \quad (41)$$

The potential (39) is of course also valid at strong coupling and we can enquire as to the fate of these non-BPS states in this region. In fact we find that in the upper-half plane the potential vanishes on a new curve, somewhat outside the CMS, defined by,

$$\text{Re} \left\{ \tau (1 + \bar{\tau}) \left(1 - \frac{a_D^* (a + a_D)}{|a_D| |a + a_D|} \right) \right\} = 0, \quad (42)$$

which we will call a *non-BPS curve of marginal stability* (nCMS) to emphasize the difference with the CMS curve. This curve is plotted in Fig. 2. Although we cannot calculate the mass of the composite state on this curve there are compelling arguments that the nCMS curve corresponds to the point where the mass of the composite reaches the threshold where the second inequality in (41) becomes an equality, i.e. when

$$M_{\{1,2\}}^{\text{non-BPS}}(\text{nCMS}) = |\mathcal{Z}_{\{0,1\}}| + |\mathcal{Z}_{\{1,1\}}|. \quad (43)$$

As evidence for this conclusion, note that if it is not true, then the state would either disappear from the spectrum while stable (i.e. $M_{\{1,2\}}^{\text{non-BPS}} < |\mathcal{Z}_{\{0,1\}}| + |\mathcal{Z}_{\{1,1\}}|$), or we could move to regions of the moduli space where it was genuinely unstable to decay (i.e. $M_{\{1,2\}}^{\text{non-BPS}} > |\mathcal{Z}_{\{0,1\}}| + |\mathcal{Z}_{\{1,1\}}|$). To avoid both scenarios, we require that the nCMS corresponds to the threshold condition (43). Further progress beyond the nCMS then leads to a repulsive potential and the non-BPS state consistently disappears from the spectrum.

We should emphasize that the position of the nCMS curve is not corrected by higher derivative terms in the effective action. This follows from its definition as the point where the $1/r$ potential between two BPS constituents vanishes, while the long range potential depends only on the two-derivative sector of the low energy effective theory. One consequence of this is that the threshold condition (43) also receives no corrections. Consequently, although we do not know the mass of the non-BPS bound state for generic moduli, we know its value precisely on the nCMS curve as it is given by the sum of the masses of its two BPS constituents. Moreover, near this curve the constraint (1) can always be satisfied and thus the nonrelativistic approximation is valid.

We can also deduce quite generally that the nCMS curve for a given state must lie outside the CMS curve for the corresponding BPS state related to it by conjugation of the charges of one of its constituents. In particular, for the $\{1, 2\}$ state we are considering here with constituents $\{1, 1\}$ and $\{0, 1\}$ such a conjugate state is the W^+ boson $\{1, 0\}$ with constituents $\{1, 1\}$ and $\{0, -1\}$, where we have conjugated the charge of the monopole. Denoting the respective potentials $V_{\{1,2\}}$ and $V_{\{1,0\}}$, one verifies that

$$V_{\{1,2\}} - V_{\{1,0\}} = \frac{2}{r \operatorname{Im} \tau} (|\tau|^2 - \operatorname{Re} \tau) > 0, \quad (44)$$

as follows from the positivity of $\operatorname{Im} \tau$. This result is not unexpected and is consistent with the fact that the mass of the non-BPS composite lies above its BPS bound.

Thus far, we have restricted our discussion to the upper-half u -plane, and have deduced that the state $\{1, 2\}$ dissociates on an nCMS curve, as shown in Fig. 2. The state $\{1, 2\}$ also exists in the lower-half u -plane, for large $|u|$, and we can again ask what happens as we approach the strong coupling domain. In the lower half-plane the only “localized” constituents are $\pm\{0, 1\}$ and $\pm\{-1, 1\}$, which are necessarily the lightest in the strong coupling region. In terms of these constituents, the state $\{1, 2\}$ is actually a four-particle configuration comprised as follows:

$$\{1, 2\} \Longleftrightarrow \{1, -1\} + 3\{0, 1\}. \quad (45)$$

This configuration can in principle dissociate in a more complicated manner than the two-particle systems we have considered thus far. However, the problem is still tractable following the procedure discussed above. Noting that the monopoles do not interact, while the potential for $\{1, -1\}$ and $\{0, 1\}$ is attractive outside, and repulsive inside, the CMS in the lower-half plane, we find that the system can be separated into subgroups which minimize their energy at infinite separation only in the interior region to the CMS in the lower half plane. In fact, since the monopoles do not interact, we can understand this result most easily by noting that the wavefunction will take a simple product form with the nontrivial structure depending only on the interaction between $\{1, -1\}$ and $\{0, 1\}$. Consequently, we deduce that the composite $\{1, 2\}$ is present outside the CMS in the lower-half plane.

At first sight this seems puzzling as we have argued that the composite $\{1, 2\}$ is non-BPS and thus forms a long multiplet. Specifically, the scenario we have found here corresponds to the case where both inequalities in (41) are saturated at the same point, implying that the CMS and the nCMS coincide. Thus, precisely on the CMS we should only find BPS multiplets. It follows that the transition to the CMS must lead to some degeneracy as the

state $\{1, 2\}$ lies in a long multiplet arbitrarily close to this point. In fact, this apparent problem is resolved by noting that the composite delocalizes precisely on the CMS itself, where it is replaced by the four-particle configuration of BPS constituents, and we see that this indeed provides the required degeneracy to account for the presence of a long $\mathcal{N}=2$ multiplet.

Thus we have found a stability domain for the state $\{1, 2\}$ which is given by the exterior region to the bold curve shown in Fig. 2. The appearance of non-BPS states of this type, descending from higher charge states in $\mathcal{N}=4$ SYM, has been conjectured before in the literature from the string web construction [18]. Moreover, the existence criteria for these non-BPS states, and also their stability domains, as deduced here agrees well with earlier arguments of Bergman [18] using the string junction formalism.

Another state which we expect to be present at weak coupling is related to $\{1, 2\}$ by CP conjugation, namely the state $\{-1, 2\}$. In the lower half-plane there are only two primary constituents for this composite configuration, $\{-1, 1\} + \{0, 1\}$, and one finds a stability domain bounded by the nCMS curve. In contrast, in the upper half-plane there are four primary constituents, $\{-1, -1\} + 3\{0, 1\}$, and analysis of the possible two-body interactions indicates that the composite is stable outside the CMS. Thus the stability domain for the state $\{-1, 2\}$ is the reflection in the real u -axis of the domain for the state $\{1, 2\}$. This makes manifest a symmetry of the theory under CP conjugation of the state and a Z_2 reflection in the moduli space $u \rightarrow -u$ [9]. In other words, we can now see that the structure of CMS and nCMS curves in Fig. 2 is consistent with the underlying Z_2 symmetry $u \rightarrow -u$ of the theory, due to the fact that it acts on the basepoint used in specifying the monodromies for each singularity and hence affects the charges. Consequently, to manifest this symmetry we need to consider both composite states $\{1, 2\}$ and $\{-1, 2\}$ that are related by just such a transformation of the basepoint.

These examples provide sufficient information on the interaction channels for us to deduce the presence, and stability domains, for higher charge composite particles. In particular, we learn that all $\{n_E, n_M\}$ states with co-prime n_E and n_M – descendants of $\mathcal{N}=4$ – represent states stable in the semiclassical region. These are, of course, *not* $\mathcal{N}=2$ BPS states: they lie in long $\mathcal{N}=2$ multiplets and their masses are larger than the values specified by the central charge in their charge sector. These states are removed from the spectrum in the strong coupling regime as we cross the appropriate nCMS or CMS curve. In regions of moduli space where the latter case applies, the dissociation must respect the degeneracies implicit in passing from long to short multiplets.

For the case of pure $\mathcal{N}=2$ SU(2) SYM that we have considered in this section, the problem of constructing the spectrum of bound states was simplified somewhat by the fact that there are only two primary states – i.e. only two conserved charges, electric and magnetic – and thus deducing the existence of composite states could always be reduced to the problem of studying two-body interactions for which our long range potential is applicable. However, it's clear that in more general scenarios, say for SU(N) with $N \geq 3$, there will be more than two primary states, and thus more complex multi-body dissociations are possible for configurations carrying multiple charges. Nonetheless, one expects that at least for systems of a generic type the procedure outlined in this section will be sufficient to deduce the presence or otherwise of composite configurations.

IV. QUARK-MONOPOLE BOUND STATES IN SU(2)

For our second example we turn to $\mathcal{N}=2$ SYM with gauge group SU(2) with a single massive hypermultiplet in the fundamental representation. This theory is the simplest example which exhibits an Argyres-Douglas point [19,41] (see also [42]) where two of the singularities in the moduli space collide for a specific choice of the parameters. This phenomenon has interesting consequences for the CMS structure of the theory.

For this system, the central charges take the form

$$\mathcal{Z}_{\{n_E, n_M, s\}} = \sqrt{2} [n_E a(u, m_f) + n_M a_D(u, m_f)] + s m_f, \quad (46)$$

where m_f is the fundamental hypermultiplet mass. The quantum numbers of the fundamental fields, which we will refer to as quarks (see e.g. [4,43,44] for a discussion of the definition of the quark charges), are $n_E = \pm 1/2$, $n_M = 0$, and $s = \pm 1$. For orientation on the quark masses it is convenient to start with large m_f , i.e. $m_f \gg \Lambda_1$, where Λ_1 is the scale parameter of the theory, and we have used a chiral rotation to make m_f real and positive. Then we find that the quark states $\mathbf{n} = \pm\{1/2, 0, 1\}$ with mass $|m_f + a/\sqrt{2}|$ are heavier than the quark states $\mathbf{n} = \pm\{-1/2, 0, 1\}$ whose mass is $|m_f - a/\sqrt{2}|$. Moreover, for $u \approx m_f^2$ the lighter states $\mathbf{n} = \pm\{-1/2, 0, 1\}$ actually become massless. This is the quark singularity of the low energy effective description.

A. Semiclassical regime and the CMS

Following our discussion in the previous section, we will start by considering the weak coupling regime, $|u| \gg |\Lambda_1^2|$, where the classical approximation is valid. The features of interest are still visible in this region provided we also choose $m_f \gg \Lambda_1$ so that the quark singularity lies at weak coupling.

To construct the long range potential, we need to determine the lightest BPS states carrying electric, magnetic, and quark charge. For $m_f \gg \Lambda_1$ it is easy to verify that the lightest such states are the monopoles $\mathbf{n}^{(1)} = \{0, 1, 0\}$ and the “light” quarks $\mathbf{n}^{(2)} = \{-s/2, 0, s\}$ where $s = \pm 1$. The central charges for these states take the form

$$\mathcal{Z}_1 = \sqrt{2} a_D, \quad \mathcal{Z}_2 = s \left(m_f - \frac{a}{\sqrt{2}} \right) \quad (47)$$

and the long range interaction between them is

$$V_s = -\frac{1}{2r \operatorname{Im}\tau} \operatorname{Re} \left\{ \tau \frac{a_D^*(a - m_f \sqrt{2})}{|a_D| |a - m_f \sqrt{2}|} + s \tau \right\}. \quad (48)$$

At leading order in the weak coupling regime, where $a = \sqrt{2u}$ and $a_D = \tau a$, the potential is independent of s and takes the simple form,

$$V_s^{\text{cl}} = -\frac{1 - \operatorname{Re}(m_f/\sqrt{u})}{2r |1 - (m_f/\sqrt{u})|}. \quad (49)$$

We deduce that the potential is attractive for $\text{Re}(m_f/\sqrt{u}) < 1$ and thus there are dyonic bound states with charges $\{-s/2, 1, s\}$ for $s = \pm 1$ in this region. This conclusion agrees precisely with an analysis of the Dirac equation in the monopole background [20], which provides a useful check on our result.

On closer inspection we find that this result is actually stronger than that predicted by an analysis of the CMS conditions (34). Alignment of the central charges implies

$$\text{Re}\left(\frac{m_f}{\sqrt{u}}\right) = 1, \quad s \text{Im}\left(\frac{m_f}{\sqrt{u}}\right) \geq 0. \quad (50)$$

Thus, if we restrict our attention here to the upper-half u -plane, we see that the CMS constraint is only satisfied for $s = -1$. Hence the transition in the potential (49) at $\text{Re}(m_f/\sqrt{u}) = 1$ only corresponds to crossing a CMS for the state $\{1/2, 1, -1\}$. In contrast the constituent central charges are anti-parallel for the state $\{-1/2, 1, 1\}$ at this point. We conclude that this second state is actually a non-BPS bound state. In the classical approximation that we are using the CMS and nCMS curves for these two states coincide, and thus this result is consistent with [20]. However, once quantum corrections are included, we would expect these curves to split in accordance with corrections to the mass of the $\{-1/2, 1, 1\}$ state above its BPS bound.

This splitting can be verified from Eq. (48) by taking into account the perturbative corrections due to the running of τ , which provide the leading s dependence of the potential at weak coupling. The perturbative running coupling τ is given by

$$\tau(u) = \frac{i}{\pi} \log \frac{u}{\Lambda_0^2}, \quad \Lambda_0 = (m_f \Lambda_1^3)^{1/4}, \quad (51)$$

where Λ_0 is the scale of the effective $N_f=0$ theory. Then we find

$$V_{s=1} - V_{s=-1} = \frac{1}{r} \frac{\text{Arg } u}{\log(|u|/\Lambda_0^2)} > 0, \quad (52)$$

which indicates that the instability domain for the non-BPS bound state $\{-1/2, 1, 1\}$ is wider than that for the BPS state $\{1/2, 1, -1\}$.

B. Strong coupling and the Argyres-Douglas point

We now return to a more detailed analysis of the CMS structure for the BPS state $\{1/2, 1, -1\}$. The alignment conditions (50) have the general form,

$$\text{Im}\left[a_D^* a \left(\frac{m_f \sqrt{2}}{a} - 1\right)\right] = 0, \quad s \text{Re}\left[a_D^* a \left(\frac{m_f \sqrt{2}}{a} - 1\right)\right] \geq 0. \quad (53)$$

defining a CMS curve which by definition must pass through the quark and monopole singularities. Using explicit expressions for the period integrals (see e.g. [45, 9]), we can go beyond the weak coupling regime and construct the CMS also in the strong coupling region. We find that it forms a closed curve which approximates a cardioid, and encompasses the

quark singularity at $u \approx m^2$ and the monopole singularity at $u \approx \Lambda_0^2$. This curve is shown in Fig. 3 as a section at fixed m of the full instability domain for the state $\{1/2, 1, -1\}$ in the three dimensional space $[\text{Re}(u), \text{Im}(u), m_f]$.

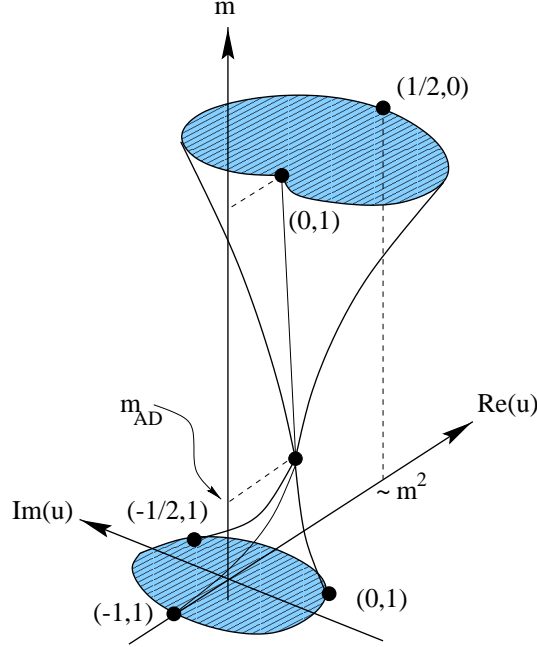


FIG. 3. The CMS curve is shown as a function of $m = m_f$ for the state $\{1/2, 1, -1\}$. For $m_f \gg m_{AD}$, the CMS takes the form of an approximate cardioid encompassing the monopole ($u \approx \Lambda_0^2$) and quark singularities ($u \approx m_f^2$). As $m_f \rightarrow m_{AD}$ this region shrinks to a point and while for $m_f = 0$ it becomes the massless CMS curve $\text{Im}(a_D^* a) = 0$. The singularities are labeled with their charges.

Thus far we have considered the regime where $m_f \gg \Lambda_1$. It is clear that the picture remains qualitatively the same until we approach the Argyres-Douglas point at $m = m_{AD} = (3/4)\Lambda_1$ where the monopole and quark singularities coalesce and the CMS we observed above shrinks to a point. When m_f is reduced below m_{AD} the quark singularity undergoes a monodromy as we unwind the cuts and is relabeled with the charges $\{-1/2, 1, 1\}$ (not to be confused with the non-BPS state we found above). The state $\{1/2, 1, -1\}$ is still composite and on passing through the Argyres-Douglas point we find that a new CMS opens up in the strong coupling region encompassing the singularities of the monopole, which moves into the lower-half plane, and the continuation of the quark, now labeled as a $\{-1/2, 1, 1\}$ dyon, which moves into the upper-half plane. The $\{1/2, 1, -1\}$ state is excluded from the interior of this region. As $m_f \rightarrow 0$ this CMS expands and precisely when $m_f = 0$ it reaches an approximately circular form, given by $\text{Im}(a_D^* a) = 0$ (see Eq. (53)), and includes the third dyon singularity which in the upper-half plane has charges $\{-1, 1, -1\}$ (NB: these charges correspond to a specific arrangement of the cuts of $a(u)$ and $a_D(u)$ consistent with [7] for $m_f = 0$). The three singularities then form the well-known \mathbf{Z}_3 symmetric formation of the massless theory [4]. Thus we can now represent the full instability domain for the state $\{1/2, 1, -1\}$, which we present in Fig. 3, including the regime $m < m_{AD}$.

Thus, we have found that the behavior of the potential is entirely consistent with earlier

arguments in the literature regarding the BPS spectrum at weak coupling [46, 47, 20] and on flowing through the Argyres-Douglas point [7–9]. Moreover, we have also deduced the presence of non-BPS dyonic bound states with quark charge which arise in a similar manner to our discussion in the previous section. Although we have considered a rather specific example, it seems clear that similar phenomena will arise in considering other states. In the next section, we generalize somewhat further and consider higher rank gauge groups.

V. DYONS IN SU(3) SYM

New phenomena are possible when the rank of the gauge group is extended beyond 1. In particular, it will be sufficient here to focus on the simplest nontrivial example – an SU(3) gauge group. For $\mathcal{N}=2$ SYM with gauge group SU(3) and one adjoint hypermultiplet of mass m_h , the central charges are

$$\mathcal{Z}_{\{n_E^a, n_{Ma}, s\}} = \sqrt{2} [n_E^a a_a + n_{Ma} a_D^a] + s m_h. \quad (54)$$

The integral electric charges n_E^a ($a = 1, 2$) are associated with the two simple roots β_a of SU(3) while the magnetic charges n_{Ma} are associated with the corresponding dual vectors, namely the weights $\omega^a = C^{ab} \beta_b$, where C^{ab} is the inverse Cartan matrix which in our normalization takes the form,

$$C_{ab} = \beta_a \cdot \beta_b = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \quad C^{ab} = \omega^a \cdot \omega^b = \frac{4}{3} \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}. \quad (55)$$

This theory is by construction UV finite and taking $m_h = 0$ we obtain $\mathcal{N}=4$ SYM. A finite value for m_h acts as a UV cutoff in the $\mathcal{N}=2$ theory which flows in the infrared to $\mathcal{N}=2$ SYM.

A. $\mathcal{N}=4$ SYM

Let us start with $\mathcal{N}=4$ SYM, i.e. by taking $m_h = 0$. In this case $a_a = \langle \Phi \cdot \beta_a \rangle$ and $a_D^a = \tau_0 C^{ab} a_b$. As discussed in Sec. II, the usual semiclassical treatment [34, 35] makes use of magnetic charges n_M^a with an upper index³, $n_M^a = C^{ab} n_{Mb}$. The central charge then takes the form (11) which we can also write as

$$\mathcal{Z}_{\{n_E^a, n_M^a\}} = \sqrt{2} Q^a a_a, \quad Q^a = n_E^a + \tau_0 n_M^a, \quad (56)$$

where we have introduced the complex charges Q^a .

We consider the interaction between two fundamental dyons living in two different U(1) subgroups of the Cartan subalgebra, namely dyons with quantum numbers $\{(n_E^1, 0), (n_M^1, 0)\}$

³Note that the monopole $\{n_M^1 = 1, n_M^2 = 0\}$ aligned along β_1 has half integer charges n_{Ma} in the notation of the dual lattice, $\{n_{M1} = 1, n_{M2} = -1/2\}$.

and $\{(0, n_E^2), (0, n_M^2)\}$ in the notation $\mathbf{n} = \{(n_E^1, n_E^2), (n_M^1, n_M^2)\}$ (note the lifted index for magnetic charges). The central charges then take the form,

$$\mathcal{Z}_1 = \sqrt{2} Q^1 a_1, \quad \mathcal{Z}_2 = \sqrt{2} Q^2 a_2, \quad (57)$$

and the potential, which follows from the general expression (24), is in this case

$$V(r) = \frac{C_{12}}{r \operatorname{Im} \tau_0} |Q^1| |Q^2| \operatorname{Re} \left\{ \frac{Q^1 (Q^2)^*}{|Q^1| |Q^2|} - \frac{(a_1)^* a_2}{|a_1| |a_2|} \right\}, \quad (58)$$

where the element of the Cartan matrix $C_{12} = -1/2$. Both terms in the curly brackets are phase factors,

$$\frac{Q^1 (Q^2)^*}{|Q^1| |Q^2|} = e^{-i\omega_Q}, \quad \frac{(a_1)^* a_2}{|a_1| |a_2|} = e^{i\omega_V}, \quad (59)$$

where the first, ω_Q , is the relative phase of the two charges, and the second, ω_V , is the relative phase of the two vevs, $a_a = \langle \Phi \cdot \beta_a \rangle$. The sum of ω_Q and ω_V is the relative phase of the central charges introduced in Eq. (25), $\omega = \omega_Q + \omega_V$. In terms of these phases the potential is given by

$$V(r) = -\frac{1}{r \operatorname{Im} \tau_0} |Q^1| |Q^2| \sin \frac{\omega_V + \omega_Q}{2} \sin \frac{\omega_V - \omega_Q}{2}. \quad (60)$$

If the complex charges $Q^{1,2}$ are aligned, i.e. $\omega_Q = 0$, the potential (60) is negative semi-definite over the moduli space, while it is positive semi-definite if $Q^{1,2}$ are anti-aligned, i.e. $\omega_Q = \pi$.

The most interesting situation arises when $Q^{1,2}$ are not aligned. If, for example, we choose $n_M^1 = n_M^2 = 1$ then at weak coupling the phase ω_Q is small and proportional to the difference of the electric charges,

$$\omega_Q = \frac{g_0^2}{4\pi} (n_E^1 - n_E^2). \quad (61)$$

The potential is repulsive in the domain of the moduli space where $|\omega_V| < |\omega_Q|$, implying that there are no bound states with $|n_E^1 - n_E^2| > 4\pi|\omega_V|/g_0^2$. However, the potential is attractive for $|n_E^1 - n_E^2| < 4\pi|\omega_V|/g_0^2$ and consequently leads to the formation of bound states in this region.

When $\omega_V = \pm\omega_Q$ the Coulombic potential vanishes. Specifically, the relation $\omega_V = -\omega_Q$ is equivalent to the vanishing of $\omega = \omega_Q + \omega_V$ which implies that this is the CMS where the central charges $\mathcal{Z}_{1,2}$ are aligned. However, the vanishing of the potential at $\omega_V = \omega_Q$ cannot be interpreted in this way. The resolution of this point in the current system is that there is a second CMS curve related to the presence of a second central charge in the $\mathcal{N}=4$ SUSY algebra. This second charge, $\tilde{\mathcal{Z}}$, has the form

$$\tilde{\mathcal{Z}}_{\{n_E^a, n_M^a\}} = \sqrt{2} Q^a a_a^*, \quad |\tilde{\mathcal{Z}}_{\{n_E^a, n_M^a\}}| = |\mathcal{Z}_{\{-n_E^a, n_M^a\}}|, \quad (62)$$

where we have also noted that $|\mathcal{Z}|$ and $|\tilde{\mathcal{Z}}|$ are related by electric charge conjugation. As is clear from comparison of Eqs. (56) and (62) the distinction between \mathcal{Z} and $\tilde{\mathcal{Z}}$ is significant

only for gauge groups of rank greater than one. In the present context we see that the CMS for the $\tilde{\mathcal{Z}}$ central charge corresponds to $\tilde{\omega} = \omega_Q - \omega_V \rightarrow 0$, which explains the fact that we found a vanishing potential for $\omega_V = \omega_Q$.

For the bound states we are discussing, $|\mathcal{Z}| \neq |\tilde{\mathcal{Z}}|$, and thus they preserve 1/4 of the $\mathcal{N}=4$ SUSY. In recent years a considerable amount of work has been undertaken on these states in the semiclassical regime [21–28, 15, 29], and also in the context of string junctions [10, 51]. In the semiclassical regime, a more detailed picture of the dynamics has recently been obtained [21, 23, 25, 27–29] using the moduli space approximation [52], while the $1/r$ potential for the SU(3) theory was also discussed in this regime in Refs. [53, 54, 24, 15]. Our main point here is that a simple analysis of the Coulombic forces is sufficient for finding the spectrum of BPS states and also the rich structure of CMS curves which arise for higher rank gauge groups.

B. $\mathcal{N}=2$ SYM in the semiclassical range

By giving a mass m_h to the hypermultiplet fields Φ_s , we can flow to $\mathcal{N}=2$ SYM and observe how the picture above is corrected at scales well below m_h . The generic expression for the near-CMS potential, as given in (27), determines CMS curves at

$$\text{Im} \left[(n_E^{(1)a} a_a^* + n_{Ma}^{(1)} a_D^{a*}) (n_E^{(2)b} a_b + n_{Mb}^{(2)} a_D^b) \right] = 0. \quad (63)$$

The solution set of this equation is quite complex and we will not pursue an in-depth analysis of the potential here, although the behavior is not unlike that of the one-flavor SU(2) case considered above. In particular, this theory possesses an Argyres-Douglas point [19] of the same type [41].

To simplify the discussion, we will limit ourselves to the semiclassical range of large moduli. In this region the long range potential is close to the one found above in the $\mathcal{N}=4$ case modulo small corrections due to the running of the gauge couplings. This means that in the weakly coupled range of the $\mathcal{N}=2$ theory we necessarily have bound states for the same set of charges $\{n_E^a, n_M^a\}$ as in $\mathcal{N}=4$ SYM. However, as we have observed in the SU(2) examples, not all these states are BPS saturated from the point of view of $\mathcal{N}=2$ SUSY.

The example considered in detail above, involving the interaction of the states $\{(n_E^1, 0), (1, 0)\}$ and $\{(0, n_E^2), (0, 1)\}$, is sufficient to understand how the splitting of the spectrum arises. Indeed, as described above, there are two CMS curves for this system in $\mathcal{N}=4$ SYM: one, $\omega_V = -\omega_Q$, where the central charges $\mathcal{Z}_{1,2}$ are aligned; and the other, $\omega_V = \omega_Q$, where the additional $\mathcal{N}=4$ central charges $\tilde{\mathcal{Z}}_{1,2}$ were aligned. This second central charge does not lead to shortening of the $\mathcal{N}=2$ multiplets [28], and so the bound states associated with the second CMS in $\mathcal{N}=4$ are not BPS states of $\mathcal{N}=2$ SYM. This implies that, say, for positive ω_V only those states in the interval $0 < n_E^2 - n_E^1 < 4\pi\omega_V/g^2$ are BPS states of $\mathcal{N}=2$ lying in short multiplets. The states with the opposite sign of $n_E^{1,2}$ do exist as bound states in the semiclassical region when $-4\pi\omega_V/g^2 < n_E^2 - n_E^1 < 0$ [28, 15], but they are in long multiplets of $\mathcal{N}=2$ and their masses are larger than the BPS bound $|\mathcal{Z}|$, where $\mathcal{Z} = \mathcal{Z}_1 + \mathcal{Z}_2$ is the total central charge. This is consistent with the relation $|\tilde{\mathcal{Z}}| > |\mathcal{Z}|$ near the second CMS, discussed in [28].

VI. CONCLUDING REMARKS

In this paper we have argued that the generic features of the spectrum of composite BPS states in $\mathcal{N}=4$ and $\mathcal{N}=2$ SYM may be deduced purely from knowledge of the long range potential between primary constituents. Moreover, we showed that this potential was calculable given knowledge of the low energy effective theory describing the massless sector. This result is valid for generic field theoretic moduli and holds in particular in the strong coupling region due to the nonrelativistic dynamics in the near CMS regime. This limit also leads to the moduli space geometry becoming irrelevant near the CMS, where the Coulombic potential was found to have a remarkably simple form, independent of the Kähler potential.

We dwelt on several illustrative examples in the context of $\mathcal{N}=2$ SYM with gauge groups $SU(2)$ and $SU(3)$, and from this analysis we can construct a rather complete picture of the BPS spectrum in $\mathcal{N}=2$ theories. Namely, we start from the weak coupling regime where the Coulombic potential will predict a spectrum of bound states, the presence of many of which can be verified independently by semiclassical techniques. We can then follow these states on trajectories through the moduli space to strong coupling. If the states are BPS states in $\mathcal{N}=2$ then we must either encounter a CMS for the state, or it must be massless somewhere in the moduli space. These conditions are easily checked given knowledge of the central charge. In the absence of either of these scenarios, we can conclude that the semiclassical bound state was non-BPS and will generically disappear from the spectrum at curves which we called the nCMS before reaching the CMS. For special trajectories in the moduli space, it may happen that the nCMS and the CMS coincide, but we found no examples where this occurred for all possible trajectories, implying that BPS and non-BPS states are indeed distinguished via this procedure. For BPS states, this picture is closely related to similar arguments by Bilal and Ferrari [6, 7, 9]. What we have added here is that some of the working assumptions of [6, 7, 9] can be deduced directly from an analysis of the Coulombic potential.

In general, the non-BPS states predicted by the Coulombic potential in the examples of the preceding sections have a larger instability domain than the BPS states to which they are related by various conjugations of the charges. It would be interesting to explore the presence of these states in more detail. In this regard, we note that higher charge non-BPS states in $\mathcal{N}=2$ have been predicted via the analysis of string webs [51, 18]. Moreover, in these string web constructions, a version of the “s-rule” [11] is invoked to determine which states are BPS states in $\mathcal{N}=2$. In our approach this is related to the alignment condition, $\text{Re}(\tilde{\mathcal{Z}}_1 \mathcal{Z}_2) > 0$. However, it would be interesting to know if there is a constraint of this kind which is applicable locally on the Coulomb branch, as is suggested by certain string junction constructions [11]⁴.

Another issue concerns those states for which the Coulombic potential behaves as $\mathcal{O}(\omega^2)$ near the CMS. We plan to discuss these systems elsewhere [55], but we note that examples of states of this kind, accessible in the semiclassical region, have stability domains on both sides of the CMS and a vanishing equilibrium separation between the constituents at the classical

⁴We thank P. Argyres for helpful comments on these and related issues.

level. These include states with purely electric or purely magnetic charges, as mentioned in Sec. V. When the Higgs vevs are aligned these states have been argued to exist as threshold bound states in $\mathcal{N}=4$ SYM [48–50], but are not present as BPS states in $\mathcal{N}=2$ SYM [28, 56].

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